A PTAS for the horizontal rectangle stabbing problem

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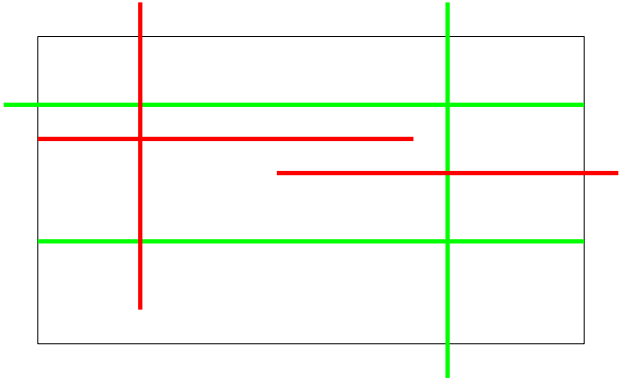
Joint work with Arindam Khan (IISc Bengaluru) and Andreas Wiese (TU of Munich)

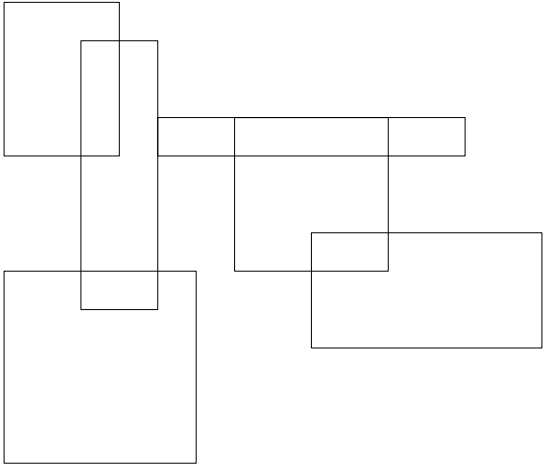
Rectangle Stabbing



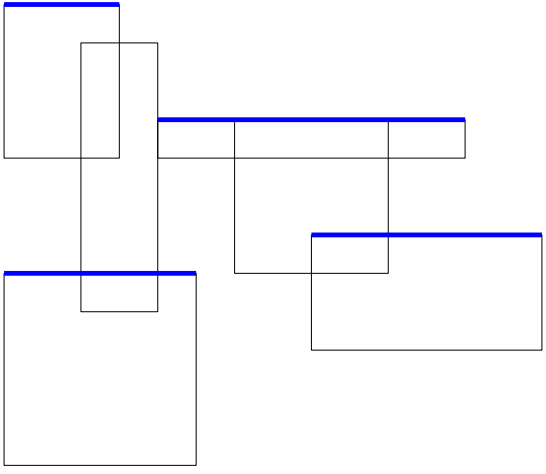
The Rectangle Stabbing Problem

Given: set of n axis-parallel rectangles $\mathcal{R}$ in the two-dimensional plane.
Goal: to compute a set of line segments $\mathcal{L}$ that stab all input rectangles.

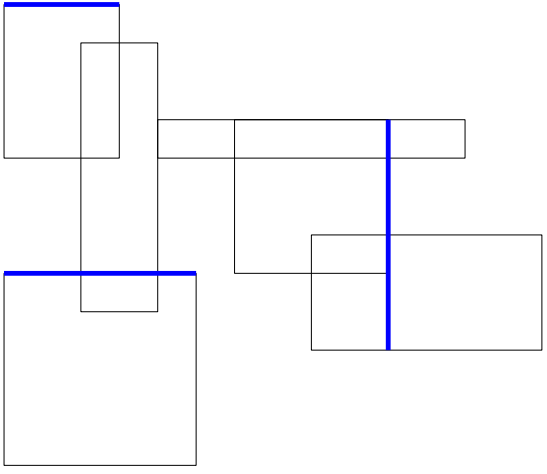
Horizontal Stabbing Problem



Horizontal Stabbing Problem

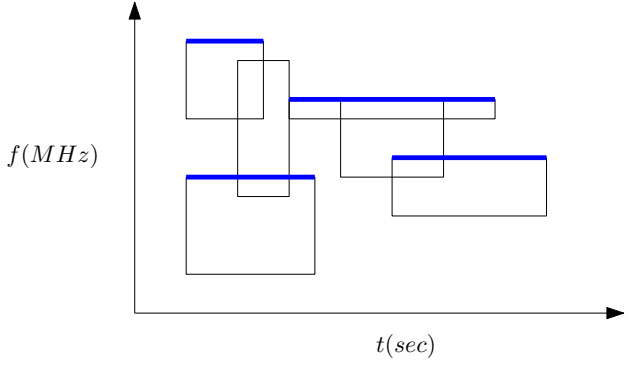


General Stabbing Problem



Motivation

- Can be viewed as a resource allocation problem.



Motivation

- Can be viewed as a resource allocation problem.
- Stabbing is a special case of weighted set cover.
- Also has applications in practical problems like VLSI circuit layout.

Past Work

- $O(1)$-**approximation** [CvDF$^+$18]: Chan, van Dijk, Fleszar, Spoerhase, and Wolff (ISAAC18) started the study of this problem and gave a constant factor approximation for general stabbing.
- $(1+\varepsilon)$-**approximation (**$n^{\text{poly}\log n}$ **time)** [EGSV21]: Recently, Eisenbrand, Gallato, Svensson, and Venzin presented a QPTAS for horizontal stabbing.

This leaves the open question of finding a PTAS for general and horizontal stabbing.

Our Results

- There is a polynomial time $(1+\varepsilon)$-approximation algorithm, for the horizontal stabbing problem.
- There is a polynomial time $(2+\varepsilon)$-approximation algorithm, for the general stabbing problem.
- There is a polynomial time $(1+\varepsilon)$-approximation algorithm, for the general stabbing problem, assuming that each input rectangle has height at least its width.
- There is a PTAS for the general square stabbing problem.

Outline of the talk

- Pre-processing
- Dynamic Programming Algorithm
- Construction of DP recursion tree

Pre-processing

Scale the widths so that $\varepsilon/n < w_i \leq 1$.
This also ensures that all the x-coordinates lie in the range $[0, n]$.

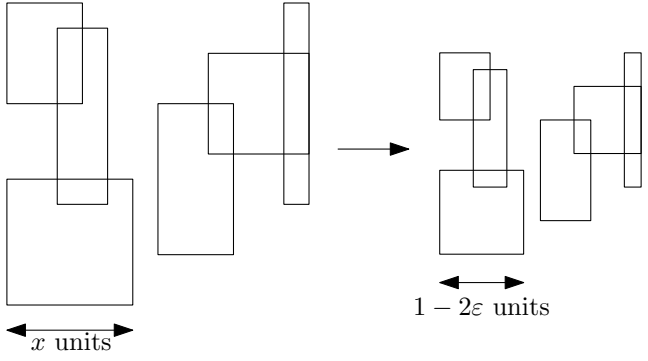


Figure: Preprocessing: Scaling

Pre-processing

Stretch the heights so that all y-coordinates lie in the range $[0, 4n^2]$.

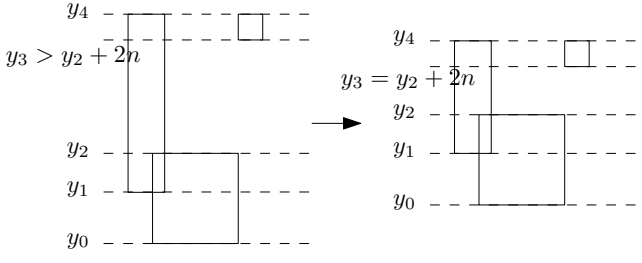


Figure: Preprocessing: Stretching

Pre-processing

Now, we discretize all the x and y-coordinates to multiples of ε/n.

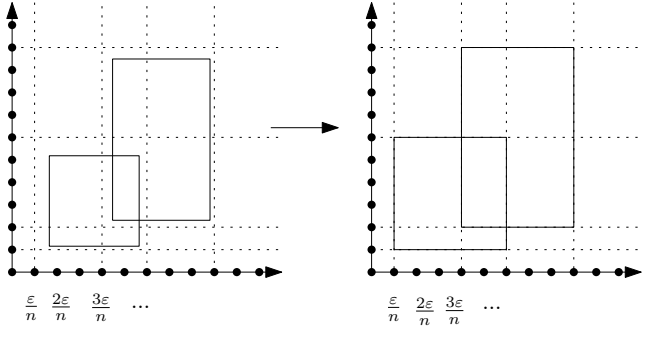


Figure: Preprocessing: Discretization

Pre-processing

Lemma

By losing a factor $(1 + O(\varepsilon))$ *in the approximation ratio, we can assume that the following properties hold:*

(i) *Width of every rectangle is in* $[\frac{\varepsilon}{n}, 1]$,

(ii) *All x-coordinates are discretized and within* $[0, n]$,

(iii) *All y-coordinates are discretized and within* $[0, 4n^2]$.

Dynamic Programming

Now we describe our DP based algorithm, which has a cell $\mathrm{DP}(S, \mathcal{L})$ for each combination of

- a rectangular region $S \subseteq [0, n] \times [0, 4n^2]$ with discretized coordinates.
- a 'small' set $\mathcal{L}$ of at most $3(1/\varepsilon)^3$ line segments with discretized coordinates.

This DP-cell corresponds to the subproblem of stabbing all rectangles contained in S which are not already stabbed by $\mathcal{L}$.

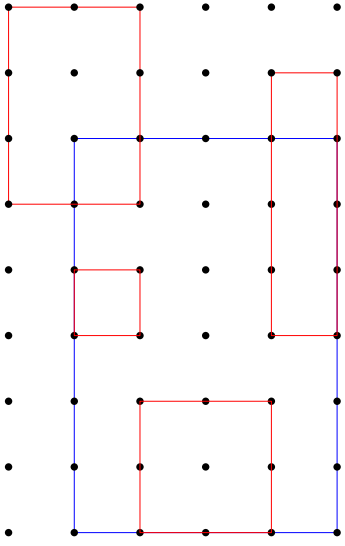


Figure: discretized cells and segments

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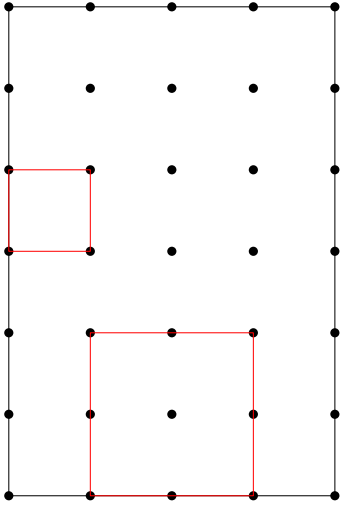

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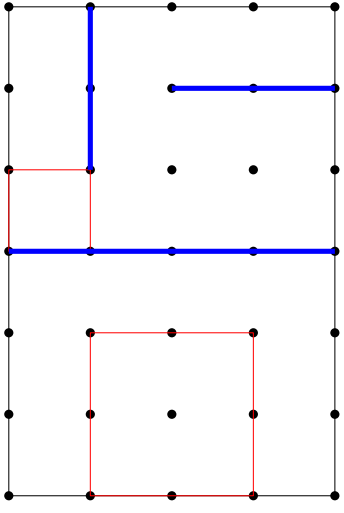

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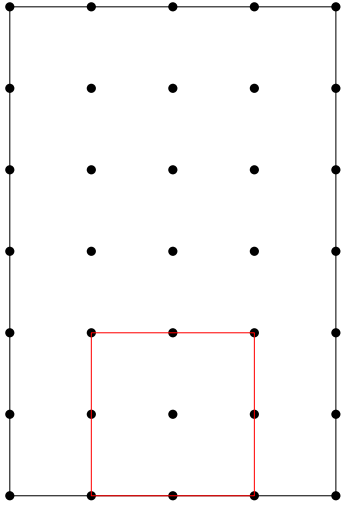
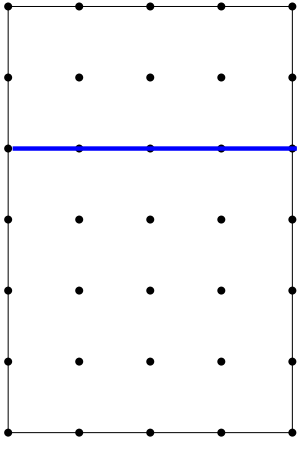


Figure: discretized cells and segments

Dynamic Programming

We do the following operations to produces a set of candidate solutions:

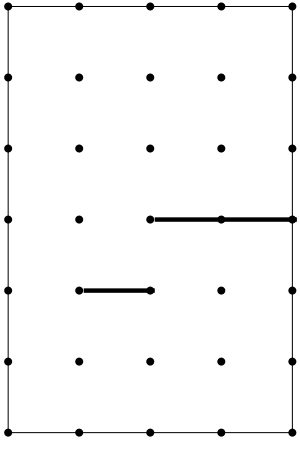
1. Trivial operation: If there is a line segment $\ell \in \mathcal{L}$ such that it divides S into 2 rectangular regions, then we add ℓ to the subproblem solution, and recurse on the 2 smaller rectangular cells.



Dynamic Programming

We do the following operations to produces a set of candidate solutions:

2. Add operation: Consider each 'small sized' set $\mathcal{L}'$, of discretized segments contained in S. We define the solution $\mathcal{L}' \cup \mathrm{SOL}(S, \mathcal{L} \cup \mathcal{L}')$ as a candidate solution.



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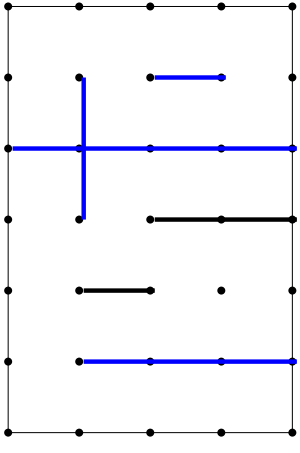
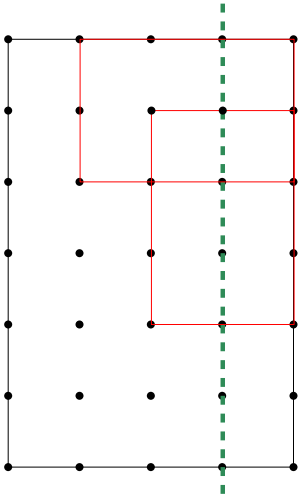


Dynamic Programming

We do the following operations to produces a set of candidate solutions:

3. Line operation: Consider each line ℓ with discretized vertical/horizontal coordinates that divides S into 2 rectangular regions, S_1 and S_2. If $\mathcal{R}_\ell$ are the rectangles (in S) stabbed by ℓ, do the following:

1. compute a $O(1)$-approximate solution $\mathcal{L}(\mathcal{R}_\ell)$, for rectangles in $\mathcal{R}_\ell$ [CvDF$^+$18].
2. produce the candidate solution as union of $\mathcal{L}(\mathcal{R}_\ell)$ and subproblem solutions of S_1 and S_2.



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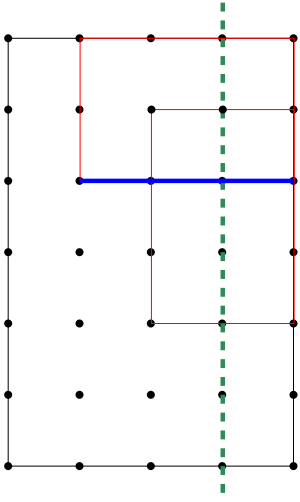

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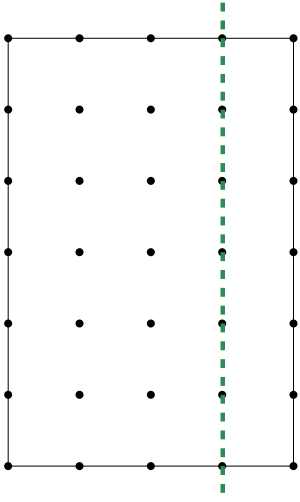

Runtime

- No. of grid points: $O_\varepsilon(\mathrm{poly}(n))$.

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- No. of DP cells: $O_\varepsilon(\mathrm{poly}(n))$.
- Possible no. of line operations: $O_\varepsilon(\mathrm{poly}(n))$.

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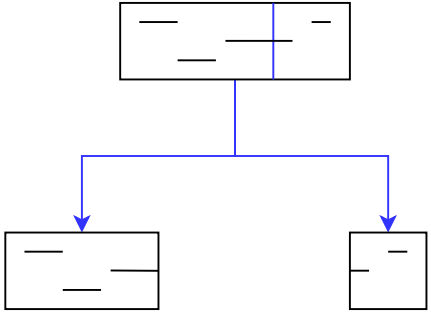
- No. of DP cells: $O_\varepsilon(\mathrm{poly}(n))$.
- Possible no. of line operations: $O_\varepsilon(\mathrm{poly}(n))$.
- Possible no. of add operations: $O_\varepsilon(\mathrm{poly}(n))$ (note that trivial operations on any segment can be charged to the corresponding add operation on the same segment).

This brings the total runtime of our algorithm to $O_\varepsilon(\mathrm{poly}(n))$ as required.

DP Decision Tree

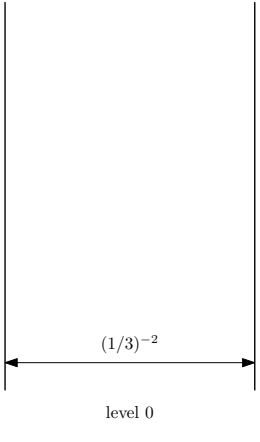
We define a DP decision tree, as a tree that describes any valid run of a recursive algorithm that breaks down a given subproblem into smaller subproblems using one of the 3 valid DP operations.



Tree construction

We define now a DP-decision-tree for which $cost(\overline{\mathrm{SOL}}(S, \mathcal{L})) \leq (1+\varepsilon)\mathrm{OPT}$.

- We start by defining a hierarchical grid of vertical lines. The grid lines have levels. For each level $j \in \mathbb{N}_0$, there is a grid line $\{a + k \cdot \varepsilon^{j-2}\} \times \mathbb{R}$ for each $k \in \mathbb{N}$, and random offset a.



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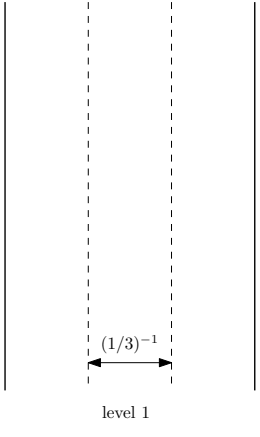

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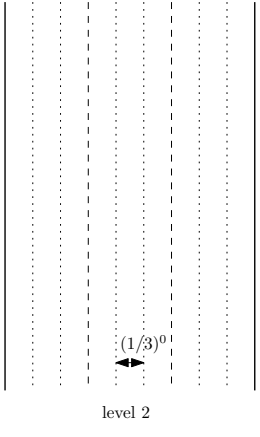

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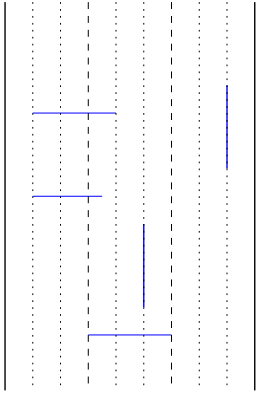
- We say that a line segment $\ell \in \mathrm{OPT}$ is of level j if $|\ell| \in (\varepsilon^j, \varepsilon^{j-1}]$.



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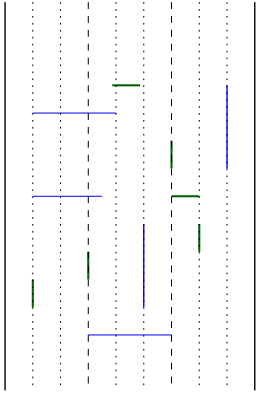

level 1

Tree construction

We define now a DP-decision-tree for which $cost(\overline{\mathrm{SOL}}(S, \mathcal{L})) \leq (1+\varepsilon)\mathrm{OPT}$.

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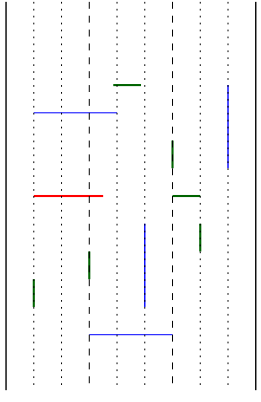
- We say that a line segment $\ell \in \mathrm{OPT}$ is of level j if $|\ell| \in (\varepsilon^j, \varepsilon^{j-1}]$.
- We say that a line segment of some level j is well-aligned if both its end-points lie on a vertical(/imaginary horizontal) grid line of level $j+3$.



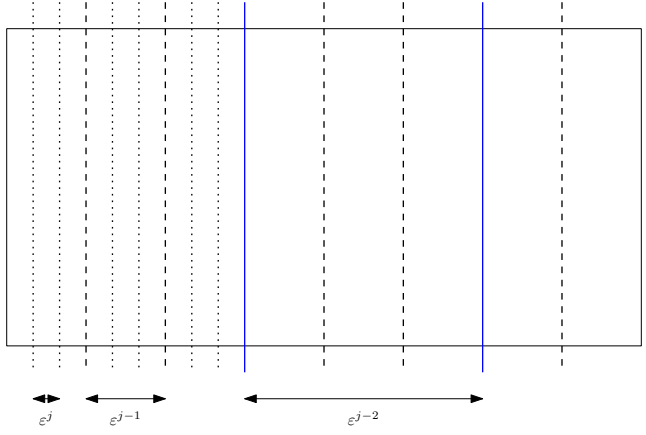
Tree construction

Lemma

By losing a factor $1 + O(\varepsilon)$*, we can assume that each line segment* $\ell \in \mathrm{OPT}$ *is well-aligned.*

Tree construction
(at level j)

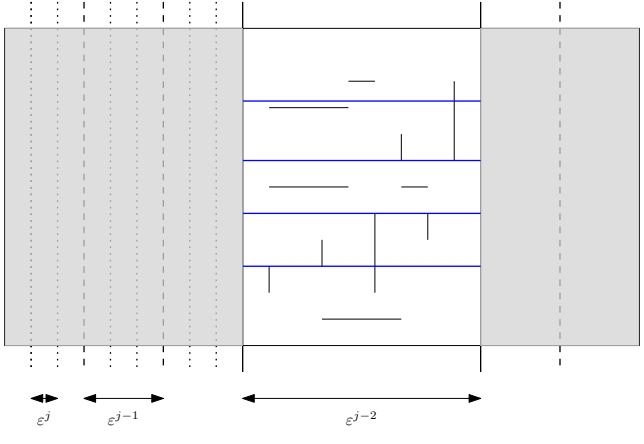
At some level j, we do line operations along the grid lines of that level.



Tree construction

(at level j)

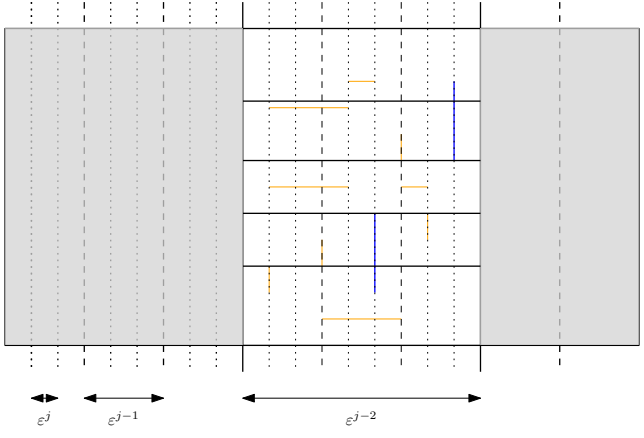
In a subproblem thus created, if there are more than ε^{-3} end points of segments from OPT of level j, do a horizontal line operation to divide into further cells.



Tree construction

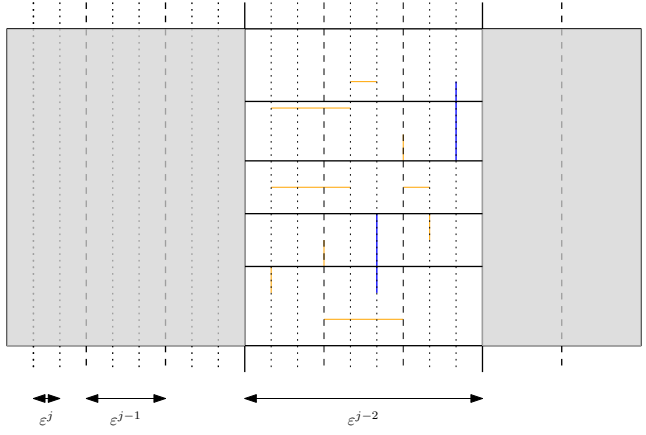
(at level j)

This is where the magic happens! Do trivial operations, and line operations on segments from level $j-2$.



Tree construction
(at level j)

Do appropriate add operations.



Random offset gives good solution

Lemma

There is a choice for the offset a, such that the solution $\overline{\mathrm{SOL}}([0, n] \times [0, 4n^2], \emptyset)$ in T has a cost of at most $(1 + O(\varepsilon))\mathrm{OPT}$.

Theorem

There is a polynomial time $(1 + \varepsilon)$-approximation algorithm, for the general stabbing problem, assuming that each input rectangle has height greater than its width.

Results

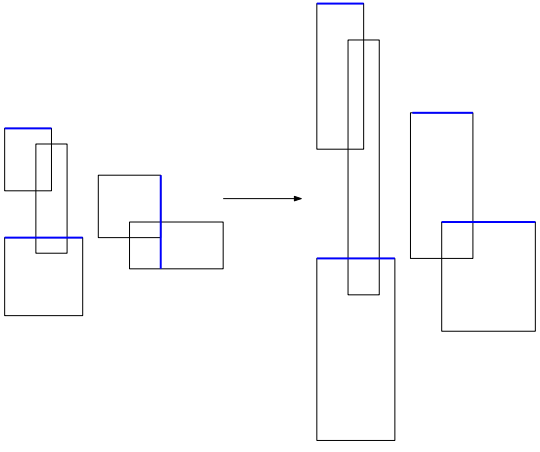
Corollary

There is a PTAS for the general square stabbing problem.

Results

Corollary

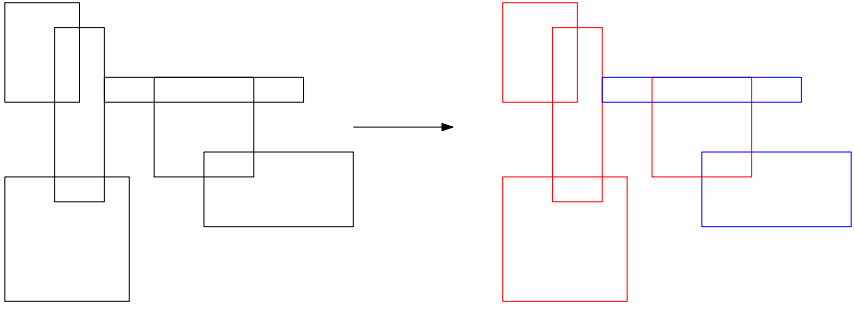
There is a polynomial time $(1+\varepsilon)$-approximation algorithm, for the horizontal stabbing problem.



Results

Corollary

There is a polynomial time $(2+\varepsilon)$-approximation algorithm, for the general stabbing problem.



Open Questions

- Is there a PTAS possible for the general rectangle stabbing problem? Can ideas from our DP be reused?
- Does Cardinality or Constrained Stabbing admit a constant factor approximation?

PTAS for δ-large Rectangles

- Preprocess the input (as before) incurring a factor $(1 + \varepsilon)$ loss.
- Do vertical line operations along level 0 grid lines, and horizontal line operations to divide instance into smaller cells.
- There are at most $O_{\varepsilon,\delta}(1)$ 'large' segments in a cell. Guess them by enumeration.
- The remaining instance can now be partitioned into two disjoint instances with $h_i \geq w_i$ and $w_i > h_i$.

PTAS for δ-large Rectangles

Theorem

For Gen-Stabbing *with δ-large rectangles, there is a $(1+\varepsilon)$-approximation algorithm with a running time of $(n/\varepsilon)^{O(1/\delta\varepsilon^3)}$.*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